§. INTEGRATION BY PARTS

.1. Integration by Parts. Examples

Let the functions $f(x), g(x), f'(x), g'(x)$ be continuous on an interval $[a,b]$. Then

$$\int f(x)g'(x)dx = \int f(x)dg(x) =$$
$$= f(x)g(x) - \int g(x)f'(x)dx$$

or

(1) $$\int udv = uv - \int vdu,$$

where $u = f(x)$, $dv = g'(x)dx$ are the parts of the integrand.

The formula (1) applies for integrals such as:

1) $\int P_n(x)e^{kx}dx$, $\int P_n(x)\sin kxdx$, $\int P_n(x)\cos kxdx$,

where $P_n(x)$ is a polynomial of x of power n and k is a constant. The evaluation of such kind of integrals includes:

a) the variable **u** to be the polynomial, i.e. $u = P_n(x)$;

b) applying of formula (1) n-times.

2) $\int P_n(x)\ln xdx$, $\int P_n(x)\arcsin xdx$, $\int P_n(x)\arccos xdx$,

$\int P_n(x)\mathrm{arctg}xdx$, $\int P_n(x)\mathrm{arcctg}xdx$,

where $P_n(x)$ is a polynomial of x of power n. The evaluation includes:

a) $u = f(x) \neq P_n(x)$;

b) applying of formula (1) .

3) $\int e^{ax}\cos bxdx$, $\int e^{ax}\sin bxdx$

where a, b are any constants. The evaluation includes:

a) $u = \cos bx$ or $u = \sin bx$;

b) applying of formula (1) two times.

Integration by Parts

Maple commands.

It is easy to see the process of evaluating integrals by parts with the subprogram

```
>with(student):
```

and the command

```
>intparts(A,u));
```

where **A** is the integral

```
>A:=Int(f,x);
```

and the function **u** is defined by the rules 1) ÷ 3).

It is also good to use the command **simplify** for clearness of the results.

Example. Evaluate the integral:

$$I_1 = \int (6x-3)\, sin\, 2x dx.$$

Mathematical Solution.

$$\left| \begin{array}{l} 6x-3=u \Rightarrow du=6dx \\ \int sin\,2xdx=\int dv \Rightarrow v=-\frac{1}{2}cos\,2x \end{array} \right| \Rightarrow$$

$$I_1 = \int \underbrace{(6x-3)}_{u} d\underbrace{\left(-\frac{1}{2}cos\,2x\right)}_{v} =$$

$$= \underbrace{(6x-3)}_{u}\underbrace{\left(-\frac{1}{2}cos\,2x\right)}_{v} - \int \underbrace{\left(-\frac{1}{2}cos\,2x\right)}_{v} d\underbrace{(6x-3)}_{u} =$$

$$= (6x-3)\left(-\frac{1}{2}cos\,2x\right) - \int -\frac{1}{2}cos\,2x.6dx =$$

$$= -\frac{6x-3}{2}cos\,2x + \frac{3}{2}sin\,2x + C.$$

Solution with Maple.

```
>I[1]:=int((6*x-3)*sin(2*x),x);
```

$$I_1 := \frac{3}{2}sin(2x) - 3x\,cos(2x) + \frac{3}{2}cos(2x)$$

Detailed Solution with Maple.

```
>with(student):
```

```
>A:=Int((6*x-3)*sin(2*x),x);
```

$$A := \int (6x-3)\,sin(2x)\,dx$$

```
>J:=simplify(intparts(A,6*x-3));
```

$$J := -\frac{6x-3}{2}cos(2x) + 3\int cos(2x)\,dx =$$

The evaluation of the integral $J_1 = 3\int cos(2x)dx$ is as usual:

```
>J[1]:=int(3*cos(2*x),x);
```

$$J_1 := \frac{3}{2}sin(2x)$$

The solution is:

$$I_1 = -\frac{6x-3}{2}cos(2x) + J_1 =$$

$$= -\frac{6x-3}{2}cos(2x) + \frac{3}{2}sin(2x) + C\,.$$

Example. Evaluate the integral:

$$I_2 = \int (x+2)\cos x dx\,.$$

Mathematical Solution.

$$I_2 = \int (x+2)\,d(\sin x) = (x+2)\sin x - \int \sin x d(x+2) =$$

$$= (x+2)\sin x - \int \sin x dx = (x+2)\sin x + \cos x + C\,.$$

Detailed Solution with Mape.

```
>with(student):
>A:=Int((x+2)*cos(x),x);
```

$$A := \int (x+2)\cos(x)\,dx$$

```
>J:=simplify(intparts(A,x+2));
```

$$J := (x+2)\sin(x) - \int \sin(x)\,dx$$

```
>J[1]:=int(sin(x),x);
```

$$J_1 := -\cos(x)$$

The solution is:

$$I_2 = (x+2)\sin x - J_1 = (x+2)\sin x + \cos x + C\,.$$

Solution with Mape (checking).

```
>A:=int((x+2)*cos(x),x);
```

Example. Evaluate the integral:

$$I_3 = \int x^2 \sin x dx$$

Mathematical Solution.

$$\left.\begin{cases} u = x^2 \Rightarrow du = 2xdx \\ \int dv = \int \sin x dx \Rightarrow v = -\cos x \end{cases}\right\} \Rightarrow$$

$$I_3 = \int \underbrace{x^2}_{u} \underbrace{\sin x dx}_{dv} = \int \underbrace{x^2}_{u} d\underbrace{(-\cos x)}_{v} =$$

$$= -x^2 \cos x + \int \cos x d\left(x^2\right) = -x^2 \cos x + \int \cos x . 2xdx =$$

$$\left.\begin{cases} u = x \Rightarrow du = dx \\ \int dv = \int \cos x dx \Rightarrow v = \sin x \end{cases}\right\} \Rightarrow$$

$$I_3 = -x^2 \cos x + 2\int \underbrace{x}_{u} d\underbrace{(\sin x)}_{v} =$$

$$= -x^2 \cos x + 2\left[x \sin x - \int \sin x dx\right] =$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C .$$

Detailed Solution with Maple.

```
>with(student):
>A:=Int(x^2*sin(x),x):
>J:=simplify(intparts(A,x^2));
```

$$J := -x^2 \cos(x) + \int \cos(x) 2x dx$$

```
>B:=2*Int(cos(x)*x,x):
>J[1]:=simplify(intparts(B,x));
```

$$J_1 : 2x \sin x - 2\int \sin x dx$$

```
>J[2]:=int(2*sin(x),x):
```

$$J_2 := -2\cos(x)$$

The solution is:

$$I_2 = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

Example. Evaluate the integral:

$$I_4 = \int e^{-2x} \cos x dx .$$

Mathematical Solution.

$$\left.\begin{cases} u = e^{-2x} \Rightarrow du = -2e^{-2x}dx \\ \int dv = \int \cos x dx \Rightarrow v = \sin x \end{cases}\right\} \Rightarrow$$

$$I_4 = \int \underbrace{e^{-2x}}_{u} \underbrace{\cos x dx}_{dv} = \int \underbrace{e^{-2x}}_{u} d\underbrace{\left(\sin x\right)}_{v} =$$

$$= e^{-2x}\sin x - \int \sin x d\left(e^{-2x}\right) = e^{-2x}\sin x + 2\int \underbrace{\sin x}_{v} \underbrace{e^{-2x}}_{u} dx =$$

$$\left.\begin{cases} u = e^{-2x} \Rightarrow du = -2e^{-2x}dx \\ \int dv = \int \sin x dx \Rightarrow v = -\cos x \end{cases}\right\} \Rightarrow$$

$$I_4 = e^{-2x}\sin x + 2\int \underbrace{e^{-2x}}_{u} d\underbrace{\left(-\cos x\right)}_{v} =$$

$$= e^{-2x}\sin x - 2e^{-2x}\cos x + 2\int \cos x de^{-2x} =\backslash$$

$$= e^{-2x}\sin x - 2e^{-2x}\cos x - 4\int e^{-2x}\sin x dx \Rightarrow$$

$$I_4 = e^{-2x}\sin x - 2e^{-2x}\cos x - 4I_4 \Rightarrow$$

$$I_4 = \frac{1}{5}e^{-2x}\left(\sin x - 2\cos x\right) + C.$$

Detailed Solution with Maple.

```
>with(student):
>A:=Int(exp(-2*x)*cos(x),x):
>J:=simplify(intparts(A,exp(-2*x)));
```

$$J := e^{(-2x)}\sin(x) + 2\int e^{(-2x)}\sin(x)dx$$

Then

$$J := e^{(-2x)}\sin(x) + J_1.$$

Evaluation J_1:

```
>B:=2*Int(exp(-2*x)*sin(x),x):
>J[1]:=simplify(intparts(B,exp(-2*x)));
```

$$J_1 := -2e^{(-2x)}\cos(x) - 4\int e^{(-2x)}\sin(x)dx$$

Thus,

$$J := e^{(-2x)}\sin(x) - 2e^{(-2x)}\cos(x) - 4J\,.$$

It follows that

$$J := \frac{1}{5}e^{(-2x)}\left(\sin x - 2\cos x\right) + C\,.$$

Solution with Maple (checking).

```
>J:=int(exp(-2*x)*cos(x),x);
```

Example. Evaluate the integral:

$$I_5 = \int \frac{x^2}{x^2 + a^2}\,dx\,.$$

Remark. This integral is called "100 000" in connection with the fact that 100 000 students have not passed the mathematical exam on integrals because of this integral.

Mathematical Solution.

$$\left\{\begin{array}{l} u = x \Rightarrow du = dx \\ \int dv = \int \frac{x}{\left(x^2 + a^2\right)^2}dx = \frac{1}{2}\int \frac{1}{\left(x^2 + a^2\right)^2}d\left(x^2 + a^2\right) \\ \Rightarrow v = -\frac{1}{2\left(x^2 + a^2\right)} \end{array}\right\} \Rightarrow$$

$$I_5 = \int \underbrace{x}_{u}\, d\underbrace{\left(-\frac{1}{2\left(x^2 + a^2\right)}\right)}_{v} =$$

$$= -\frac{x}{2\left(x^2 + a^2\right)} + \frac{1}{2}\int \frac{1}{\left(x^2 + a^2\right)}dx =$$

$$= -\frac{x}{2\left(x^2 + a^2\right)} + \frac{1}{2a}\,arctg\,\frac{x}{a} + C\,.$$

<u>Detailed Solution with Maple.</u>

```
>with(student):
>A:=Int(exp(-2*x)*cos(x),x):
>J:=simplify(intparts(A,exp(-2*x)));
```

<u>???</u>

.2. Selftraining Problems

1) Evaluate the integrals:

$$I_6 = \int arctg\sqrt{2x-1}dx,$$

$$I_7 = \int ln(4x^2+1)dx,$$

$$I_8 = \int x^2 e^x dx,$$

$$I_9 = \int e^{3x} sin 2x dx,$$

$$I_{10} = \int sin\, ln\, x dx.$$

<u>Mathematical Solution of</u> I_6.

$$\left| \begin{array}{l} arctg\sqrt{2x-1} = u \Rightarrow \\ du = \frac{1}{1+\left(\sqrt{2x-1}\right)^2} \cdot \frac{2}{2\sqrt{2x-1}} dx = \frac{1}{2x\sqrt{2x-1}} dx \\ \int dx = \int dv \Rightarrow v = x \end{array} \right| \Rightarrow$$

$$I_6 = x.arctg\sqrt{2x-1} - \int x \cdot \frac{1}{2x\sqrt{2x-1}} dx =$$

$$= xarctg\sqrt{2x-1} - \int \frac{dx}{2\sqrt{2x-1}} =$$

$$= xarctg\sqrt{2x-1} - \frac{1}{2}\sqrt{2x-1} + C$$

<u>Solution with Maple of</u> I_6.

```
>I[6]:=int(arctan(sqrt(2*x-1)),x);
```

$$I_6 := \frac{1}{2}(2x-1)arctg\left(\sqrt{2x-1}\right) - \frac{1}{2}\sqrt{2x-1} + \frac{1}{2}arctg\left(\sqrt{2x-1}\right)$$

Solutions with Maple.

```
>I[7]:=int(ln(4*x^2+1),x);
```

$$I_7 := x\,ln\left(4x^2+1\right) - 2x + arctan(2x)$$

```
>I[8]:=int(x^2*exp(x),x);
```

$$I_8 := x^2e^x - 2xe^x + 2e^x$$

```
>I[9]:=int(exp(3*x)*sin(2*x),x);
```

$$I_9 := -\frac{2}{13}e^{(3x)}cos(2x) + \frac{3}{13}e^{(3x)}sin(2x)$$

```
>I[10]:=int(sin(ln(x)),x);
```

$$I_{10} := -\frac{1}{2}cos(ln(x))\,x + \frac{1}{2}sin(ln(x))\,x$$

2) Evaluate the integrals:

$$I_{11} = \int x^2 \sin x dx,$$

$$I_{12} = \int x \ln x dx,$$

$$I_{13} = \int x^2e^{5x}dx,$$

$$I_{14} = \int x^5e^{x^2}dx,$$

$$I_{15} = \int e^{3x}(\sin 2x - \cos 2x)dx,$$

$$I_{16} = \int x(arctgx)^2 dx,$$

$$I_{17} = \int \frac{x.arctgx}{\sqrt{1+x^2}}dx,$$

$$I_{18} = \int \frac{x^2arctgx}{1+x^2}dx,$$

$$I_{19} = \int \frac{arcsin\sqrt{x}}{\sqrt{1-x}}dx,$$

$$I_{20} = \int x^4e^{3x}\sin x dx,$$

$$I_{21} = \int ln\left(x^2 + 2\right)dx,$$
$$I_{22} = \int x^2\, ln(1 + x)\,dx.$$

.3. Selfcontrol Test

Evaluate the integrals:

$$I_{23} = \int arctg\sqrt{x}dx,$$
$$I_{24} = \int e^{2x} co\,s\,xdx,$$
$$I_{25} = \int ln^2 xdx,$$
$$I_{26} = \int \frac{ln\,x}{x^3}dx.$$

.4. Selfcontrol Questions

1) Explain the idea of integration by parts.

2) How you can definite the function **`u`** for integration by parts?

3) Explain the meant of the *Maple* commands:

```
with(student):
A:=Int(f,x);
intparts(A,u));
simplify
```